Parabolic Hilbert schemes on singular curves and representation theory

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(joint w. T. Simental
M. Vazirani)

$C = \{f(x,y) = 0\} \subset \mathbb{C}^2$

$\mathcal{O}_C$ = ring of fns on C $= \dfrac{\mathbb{C}[x,y]}{(f(x,y))}$

$\mathcal{O}_{C,0}$ = local ring at 0 $= \dfrac{\mathbb{C}[[x,y]]}{(f(x,y))}$

Hilbert scheme of points on C:

$\mathrm{Hilb}^n C = \{\text{ideals } I \text{ in } \mathcal{O}_C : \dim \mathcal{O}_C/I = n\}$

$\mathrm{Hilb}^n (C,0) = \{\text{ideals in } \mathcal{O}_{C,0} : \dim \mathcal{O}_{C,0}/I = n\}$

Goal: understand the geometry of $\mathrm{Hilb}(C)$ by means of rep. theory

Ex C smooth $= \{y = 0\}$

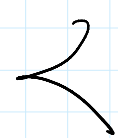
$\mathcal{O}_C = \mathbb{C}[x]$ $\quad \mathcal{O}_{C,0} = \mathbb{C}[[x]]$

$\mathrm{Hilb}^n C = \{\text{principal ideals } (f) : f \text{ is a degree } n \text{ polynomial}\}$

$= \mathbb{C}^n$

$\mathrm{Hilb}^n (C,0) = \{(x^n)\} = \{\text{point}\}$.

Ex $C = \{x^2 = y^3\}$ cusp sing.



$x = t^3 \quad y = t^2$

$\mathcal{O}_{C,0} = \dfrac{\mathbb{C}[[x,y]]}{(x^2 - y^3)} = \mathbb{C}[[t^2, t^3]]$

$\mathrm{Hilb}^n (C,0)$

(x^2-y^3)

$\mathrm{Hilb}^n(C,0)$:

$n=0 \qquad \{\mathcal{O}_{C,0}\} \qquad pt$

$n=1 \qquad \{\mathfrak{m}\} \qquad pt$

$n=2 \qquad \{(t^2+\lambda t^3),\ \lambda\in\mathbb{C}$
$\qquad\qquad (t^3,t^4)\} \qquad \mathbb{CP}^1 = \mathrm{Hilb}^2(C,0)$

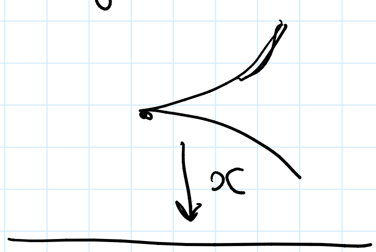
$n\geq 2 \qquad \{(t^n+\lambda t^{n+1}),$
$\qquad\qquad (t^{n+1},t^{n+2})\} \qquad \mathrm{Hilb}^n(C,0)=\mathbb{CP}^1$ for $n\geq 2$.

Remarks: (a) $\mathrm{Hilb}^n C$ is determined by
$\mathrm{Hilb}^n(C^{sm}) = S^n(C^{sm})$ smooth part
and local $\mathrm{Hilb}^n(C,p)$ where $p=$ singular pt.

(b) If $C=$ locally irreducible, with planar singularities
then $\mathrm{Hilb}^n(C,0)$ stabilize for $n\gg 0$
[Altmann, Kleiman, Iarrobino]

(c) In general, $\mathrm{Hilb}^n(C)$ and $\mathrm{Hilb}^n(C,0)$ are very singular!

We will need some cousins of Hilb^n which depend on a projection to some line



$$\mathrm{PHilb}^{k,n+k} = \{\mathcal{O}_C \supset I_k \supset I_{k+1} \supset \dots \supset I_{k+n} = x I_k\}$$

parabolic Hilbert scheme

where $I_s=$ ideal in $\mathcal{O}_C$ and $\dim \mathcal{O}_C/I_s = s$.

$n=$ degree of the projection $=$

n $= \dim I_k/xI_k = \dim \mathcal{O}_C/x$

n – degree of the projection –

$$= \dim I_k / xI_k = \dim \mathcal{O}_C / x\mathcal{O}_C$$

$$\mathrm{PHilb}^{\gamma, x} = \{ \mathcal{O}_C \supset \overbrace{J^0 \overset{\gamma_1}{\supset} J^1 \overset{\gamma_2}{\supset} \cdots \supset xJ^0}^{n} \}$$

where $\dim J^{s-1}/J^s = \gamma_s$ for some composition $\gamma = (\gamma_1 \ldots \gamma_r)$

$\gamma_1 + \ldots + \gamma_r = n.$

$$\mathrm{CPHilb}^{r,x} = \bigsqcup_{\substack{\gamma \text{ has} \\ r \text{ parts}}} \mathrm{PHilb}^{\gamma, x}$$

"compositional parabolic Hilbert scheme".

<u>Thm</u> [G., Simental, Vazirani]

$C = \{x^m = y^n\}$ $\gcd(m,n) = 1$

$\mathbb{C}^*$ action $(x,y) \to (t^m \cdot x, t^n \cdot y)$

this extends to $\mathbb{C}^*$ action on <u>all</u> Hilbert schemes in question.

(a) $\bigoplus_{k=0}^{\infty} H_*^{\mathbb{C}^*}(\mathrm{PHilb}^{k, n+k}(C))$

has an action of the rational Cherednik algebra with parameter $c = m/n$

(b) $\bigoplus_{k=0}^{\infty} H_*^{\mathbb{C}^*}(\mathrm{Hilb}^k(C))$ has an action of the <u>spherical</u> rational Cherednik algebra $c = \frac{m}{n}$.

(c) There is a version of these results at "$m \to \infty$"

$\{x^m = y^n\}$ "$\longrightarrow$" $\{y^n = 0\}$ ← non-reduced curve!

(line with multiplicity n).

Results in (a) and (b) also work in this case, and there is an action of rational Cherednik algebra "at $c = \infty$" in $H_*^{\mathbb{C}^*}(\mathrm{Hilb}(\{y^n = 0\}))$

(d) There is an action of quantized Gieseker algebra

(d) There is an action of quantized Gieseker algebra $\mathcal{A}_c(n,r)$ with $c = \frac{m}{n}$ in $H_*^{\mathbb{C}^*}(\mathbb{C}P\,\mathrm{Hilb}^{r,\frac{y}{n}})$ note the swap $x \leftrightarrow y$.

And in all these cases we can identify the representations of the corresponding algebras explicitly.

What are all these algebras?

(a) Rational Cherednik algebra $H_c(n)$

generated by $x_1 \ldots x_n, y_1, \ldots y_n, \mathbb{C}[S_n]$

with relations: $[x_i, x_j] = 0 \quad [y_i, y_j] = 0$

S_n permutes x_i, y_i as usual.

$$[y_i, x_j] = c\,(i\ j) \quad i \neq j$$
$$[y_i, x_i] = 1 - c \sum_{j \neq i} (i\ j)$$

RHS is in $\mathbb{C}[S_n]$

Rmk In "$c = \infty$" version, ignore the term with 1

$H_c(n)$ depends on a parameter c.

(b) Spherical rational Cherednik algebra

$= eH_c(n)e$, where $e = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma$ ← idempotent in $\mathbb{C}[S_n]$

(d) $\mathcal{M}(n,r)$ = moduli space of rank r torsion free sheaves on $\mathbb{C}^2$, framed at ∞ with $c_2 = n$.

Related to moduli space of instantons on $\mathbb{R}^4$

(Atiyah, Drinfeld, Hitchin, Manin)

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$\mathcal{M}(n,r)$ = smooth algebraic variety of $\dim = 2nr$

If $r=1$, $\mathcal{M}(n,1) = \mathrm{Hilb}^n \mathbb{C}^2$

$\mathcal{A}_c(n,r) =$ quantization of $\mathcal{M}(n,r) =$

noncommutative deformation of $\mathbb{C}[\mathcal{M}(n,r)]$.

Ex $\mathcal{A}_c(n,1) = eH_c(n)e$ spherical rational Cherednik

Ex $n=1$ $\mathcal{A}_c(1,r) =$ certain quotient of $\mathcal{U}(\mathfrak{gl}(r))$

Rmk It is not known how to define $\mathcal{A}_c(n,r)$ by generators & relation.

Idea of proof: (a) We need to find interesting correspondences between different $\mathrm{PHilb}^{k,n+k}(C)$

- Recall $\mathrm{PHilb}^{k,n+k} = \{\mathcal{O}_C \supset I_k \supset I_{k+1} \supset \dots \supset I_{k+n} = xI_k\}$

Forget one of ideals in the middle and consider

$$\{\mathcal{O}_C \supset I_k \supset I_{k+1} \supset \dots \supset I_{s-1} \supset I_{s+1} \supset \dots I_{k+n}\}$$

forget I_s.

Can use pushforward & pull back along this projection to define the action of S_n

(similar to the action of S_n in $H_*(\text{flag variety})$)

- $T: \mathrm{PHilb}^{k,n+k} \longrightarrow \mathrm{PHilb}^{k+1,n+k+1}$

• $T: PHilb^{k,n+k} \longrightarrow PHilb^{k+1,n+k+1}$

$\{\mathcal{O}_C \supset I_k \supset I_{k+1} \supset \dots \supset xI_k\} \longrightarrow \{\mathcal{O}_C \supset I_{k+1} \supset \dots \supset xI_k \supset xI_{k+1}\}$

⇝ defines a map $\cancel{H}_*^{\mathbb{C}^*}(PHilb^{k,n+k}) \longrightarrow H_*^{\mathbb{C}^*}(PHilb^{k+1,n+k+1})$

• Key observation: image of T is given by flags where $\{\mathcal{O}_C \supset J_{k+1} \supset \dots \supset J_{k+n} \supset J_{k+n+1} = xJ_{k+1}\}$

where J_{k+n} is divisible by x

This is equivalent to vanishing of some section of some line bundle on $PHilb^{k+1,n+k+1}$

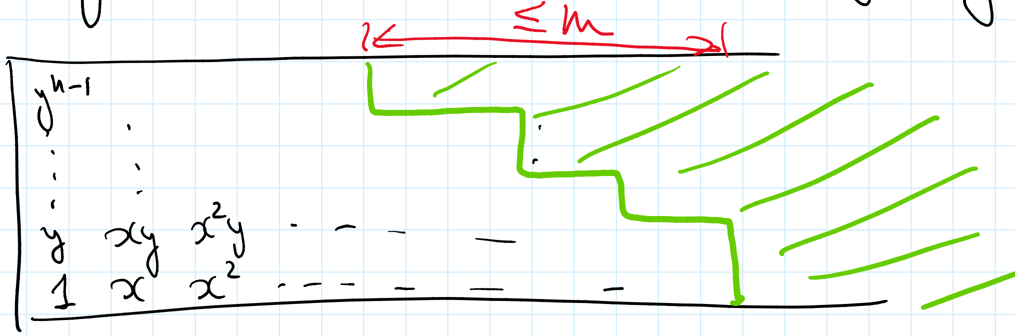
⇒ there is a Gysin map $\cancel{H}_*(PHilb^{k+1,n+k+1}) \to H_*(PHilb^{k,n+k})$

One can use $\cancel{T}$, Gysin $= \lambda$, action of S_n to define the action of Cherednik algebra

$\tau = (1 \dots n)x_1$ ⇝ can get x_1, and all x_i ...

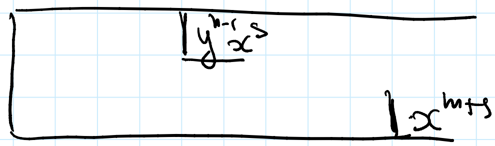
$H_*^{\mathbb{C}^*}$ has a basis given by fixed points of $\mathbb{C}^*$ action

$\mathcal{O}_{C,0} = \{x^m = y^n\}$ has a basis $\mathbb{C}[[x]]\langle 1, y, \dots, y^{n-1}\rangle$



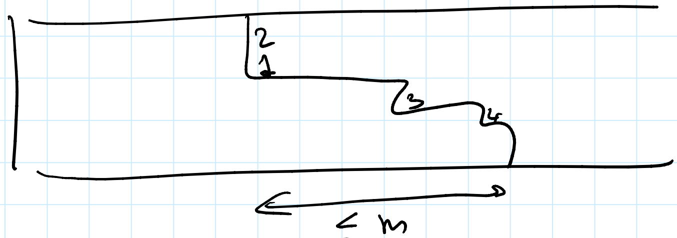
fixed points on $Hilb(C)$ = monomial ideals

fixed points in $\mathrm{Hilb}(\mathbb{C})$ = monomial ideals
= staircases in $n \times \infty$ strip with width at most m



$$y \cdot y^{n-1} x^s = y^n \cdot x^s = x^{m+s}$$

fixed points on $\mathrm{PHilb}(\mathbb{C})$ = flags of monomial ideals = staircases w. labeling at vertical runs.



Once we defined the operators geometrically, we use fixed pt basis to verify the relations & compare with a basis in rat. Cherednik rep. constructed by Griffeth.

(b) $H_*^{\mathbb{C}^*}(\mathrm{Hilb}) = \left[H_*(\mathrm{PHilb})\right]^{S_n}$

(d) To connect to $\mathcal{A}_c(n,r)$ we use a recent result of Etingof-Losev-Krylov-Simental

$$L_{\frac{m}{n}}(n,r) = \left[L_{\frac{n}{m}}(m) \otimes (\mathbb{C}^r)^m\right]^{S_m}$$

Rmk swap $x \leftrightarrow y$ matches $\frac{m}{n} \leftrightarrow \frac{n}{m}$

irrep of $\mathcal{A}_{\frac{m}{n}}(n,r)$

irrep of rat. Cherednik for $c = \frac{n}{m}$

$(G, N) \rightsquigarrow$ generalized affine Springer fiber (depends on a choice of $v \in N((t))$)

N: f.d. rep. of G

$\rightsquigarrow$ (Braverman-Finkelberg-Nakajima) Coulomb branch algebra $\mathcal{A}(G, N)$

Thm (Hilburn, Kamnitzer-Weekes) $\mathcal{A}(G,N)$ acts in H_*(affine Spr. fiber).

$G = GL(n)$ $N = \mathfrak{gl}(n) \longrightarrow$ classical affine Springer fibers

$N = \mathfrak{gl}(n) \oplus \mathbb{C}^n \longrightarrow \mathrm{Hilb}^n(C)$