Divisors on matroids and their volumes:
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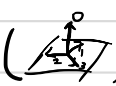
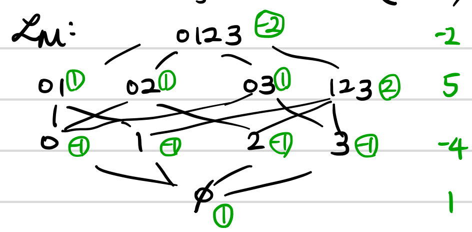
<u>Outline</u>: (A) intersection thry on matroids, (B) the volume polynom., (C) volume of a matroid.

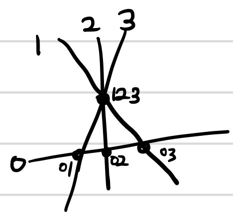
<u>(A)</u> • M (loopless, simple) matroid of rank $= r$ on n elts E. (set $d = r-1$).
• $F \subset E$ is a <u>flat</u> if $\operatorname{rk}(F \cup \{x\}) > \operatorname{rk} F \quad \forall x \in E \setminus F$. $\rightsquigarrow$ lattice of flats $\mathcal{L}_M$. $\overline{\mathcal{L}}_M = \overset{\text{cut}}{\{\emptyset, E\}}$.
• If M realizable (over $\mathbb{C}$), say $E = \{v_1, \dots, v_n\} \subset V \simeq \mathbb{C}^r \rightsquigarrow$ hyperplane arrangement $\mathcal{A}_M = \{H_i\}_{i \in E}$ in $\mathbb{P}V^* \simeq \mathbb{P}^d$ $\quad H_i = \{ f \in \mathbb{P}V^* \mid f(v_i) = 0 \}$. Let $H_F := \{ f \in \mathbb{P}V^* \mid f(v_i) = 0 \ \forall i \in F \}$.
• $\chi_M(t)$ (characteristic polynom.), $\overline{\chi}_M := \chi_M/(t-1) = \mu^0(M)t^d - \mu^1(M)t^{d-1} + \cdots \pm \mu^d(M)$
• $\mathcal{A}_M \rightsquigarrow Y_M =$ wonderful compactification: $\mathbb{P}V^*$ blown-up at all (strict transforms of) $\{H_F\}_{F \in \overline{\mathcal{L}}_M}$ starting with the highest rank F's.
• <u>Chow ring</u> of M: $A^\bullet(M)_{\mathbb{R}} := A^\bullet(Y_M)_{\mathbb{R}} = H^{2\bullet}(Y_M; \mathbb{R}) = \mathbb{R}[x_F : F \in \overline{\mathcal{L}}_M] / \langle I + J \rangle$
where $I = \langle x_F x_{F'} : F, F' \text{ incomparable} \rangle$, $J = \langle \sum_{F \ni i} x_F - \sum_{G \ni j} x_G \mid i, j \in E \rangle$.

<u>E.g.</u> $M = M($  $) = M($  $)$.

$\mathcal{L}_M$:



$\mathcal{A}_M$:



$\frac{1}{t} \cdot t(t-1)(t-1)(t-2)$

$\chi_M = 1t^3 - 4t^2 + 5t - 2$

$\overline{\chi}_M = t^2 - 3t + 2$

$A^\bullet(M)_{\mathbb{R}} = \mathbb{R}[8 \text{ var.}] / \langle I + J \rangle$.

<u>Conj.</u> (Rota) $(\mu^0, \mu^1, \dots, \mu^d)$ is log-concave $\quad \mu^i \cdot \mu^i \geq \mu^{i-1} \cdot \mu^{i+1}$ ($\Rightarrow$ unimodal)
<u>Thm</u> (Huh'12, Huh-Katz'12) $\quad \mu^i =$ intersection # of cycles in $A(M) =$ mixed vol. of convex bodies and Alexandrov-Fenchel inequality. (M realizable here). $\quad \uparrow$ Newton-Okounkov bodies
<u>Thm</u> (Adiprasito-Huh-Katz '18) still true for general matroids $\Leftarrow$ Hodge thry on $A^\bullet(M)$.

<u>(B)</u> $\quad X^d$ "nice" projective $\mathbb{C}$-variety. D a divisor. (consider $D \in H^2(X;\mathbb{Z})$ via first Chern class).
$\mathrm{Vol}_X(D) := \int_X \underbrace{D \wedge \cdots \wedge D}_{d = \dim X \text{ times}} = \mathrm{vol}\, \underline{\Delta(D)}_{\rightarrow \text{N-O body}}$. If $\{D_1, \dots, D_r\}$ generate $NS(X)_{\mathbb{R}} \subset H^2(X;\mathbb{Z})_{\mathbb{R}}$ then

$\mathrm{Vol}_X(t_1 D_1 + \cdots + t_r D_r)$ is a $\deg = d$ polynomial $VP_X \in \mathbb{R}[t_1, \dots, t_r]$ (fine print: if D nef).

<u>Rem</u> In fact, can recover $H^{2\bullet}(X;\mathbb{R})$ from VP_X.

Rem GB computation takes forever...

Thm (E.) Let $\emptyset = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq F_{k+1} = E$ chain of flats of rank $r_i := \mathrm{rk}\, F_i$, $d_1, \dots, d_k$ st $\sum_i d_i = d$. Denote $\tilde{d}_i = \sum_{j=1}^{i} d_i$. Then coeff. in VP_M of $t_{F_1}^{d_1} \cdots t_{F_k}^{d_k}$ is:

$$(-1)^{d-k} \binom{d}{d_1, \dots, d_k} \prod_{i=1}^{k} \binom{d_i - 1}{\tilde{d}_i - r_i} \mu^{\tilde{d}_i - r_i}(M|F_{i+1}/F_i)$$

(Set: $\binom{n}{m} = 0$ if $m<0$ or $m>n$ and $\mu^i(N) = 0$ if $i<0$ or $i > \mathrm{rk} N - 1$).

pf) toric variety on Bergman fan + Weisner's thm + Cover-partition axiom for flats

Prop $M \mapsto VP_M \in \mathbb{R}[t_S \mid S \subset E]$ is a valuation under matroid polytope subdivision.
Q. $\rightsquigarrow$ gen. to arbitrary Lie type?

Prop Two notable cases: $U_{n,n} \rightsquigarrow X_{A_{n-1}} \rightsquigarrow$ volumes of gen. perm. (cf. Postnikov)
$K_n \rightsquigarrow \overline{\mathcal{M}}_{0,n+1} \leftarrow$ known to be non-MDS. Q.

N.B. Better proof reflecting the str. of matroid polytope (Q.)

(C) $\mathrm{Vol}(M) := VP_M(t_F = \mathrm{rk}\, F)$ (note rk is submodular fct.).

Rem Not related to any deletion-contraction invar. (i.e. Tutte polynom.)
(Q.) Not related to volume of matroid polytope.

Thm For realizable matroids of rank $= r$ on n elts, Vol is uniquely max. at $U_{r,n}$ w/ $\mathrm{Vol}(U_{r,n}) = n^{r-1}$ (cf. $\mathbb{P}^{r-1} \overset{\deg n}{\hookrightarrow} \mathbb{P}^N$)

pf) The divisor $D = \sum (\mathrm{rk}\, F) x_F \in H^{2\bullet}(Y_M; \mathbb{R})$ is equiv. to $n\tilde{H} - E$ where $\tilde{H} = \pi^*(c_1(\mathcal{O}_{\mathbb{P}^{r-1}}(1)))$, $Y_M \xrightarrow{\pi} \mathbb{P}^{r-1}$, and E effective w/ $E = 0$ iff uniform. ($E = \sum_{\mathrm{rk} F \geq 2} |F| - \mathrm{rk}\, F$).

$\mathrm{Vol}(n\tilde{H} - E) \leq \mathrm{Vol}(n\tilde{H})$ indeed as $H^0(m(n\tilde{H} - E)) \subset H^0(m(n\tilde{H}))\ \forall m \geq 0$
(equiv. $\Delta(n\tilde{H} - E) \subset \Delta(n\tilde{H})$)

Q. True for gen. matroids? What is min. $(U_{r-2,r-2} \oplus U_{2,n-r+2})$?
$\rightsquigarrow$ Holy grail: is there $\Delta(D)$ for general matroids?