Triangulations and soliton graphs

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September 4, 2018

Outline

- Setting: The KP equation.
- Soliton graphs.
- Duality and soliton triangulations.
- Results.

The *KP* equation

- Non-linear dispersive wave equation

$$\frac{\partial}{\partial x}\left(-4\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}\right) + 3\frac{\partial^2 u}{\partial y^2} = 0.$$

- Line-soliton solutions of the KP equation model shallow-water waves with peaks localized along straight lines.

- Combinatorics of KP solitons studied in (**?**).

The Grassmannian

- *Regular* line-soliton solutions can be constructed from points in the *totally nonnegative Grassmannian.*
- For $N \leq M$, let $\mathrm{Gr}(N, M)$ be the *Grassmannian* of N-planes in M-space.
- Represent points in $\mathrm{Gr}(N, M)$ by full-rank $N \times M$ matrices, modulo row operations.

Plücker coordinates

- The $N \times N$ matrix minors give homogeneous coordinates on $\mathrm{Gr}(N, M)$, called *Plücker coordinates*.
- For I an N-element subset of $\{1, 2, \ldots, M\}$, let $\Delta_I(A)$ denote the matrix minor of A corresponding to columns indexed by I.
- The collection N-tuples indexing non-vanishing Plücker coordinates of A the *matroid* of A, denoted $\mathcal{M}(A)$.

The totally nonnegative Grassmannian

- The *totally nonnegative Grassmannian* $\mathrm{Gr}_{\geq 0}(N, M)$ is the locus of $\mathrm{Gr}(N, M)$ where all Plücker coordinates are nonnegative.
- Similarly, the *totally positive Grassmannian* $\mathrm{Gr}_{>0}(N, M)$ is the locus where all Plücker coordinates are positive.
- The combinatorics of $\mathrm{Gr}_{\geq 0}(M, N)$ was first studied in (**?**).

Notation

- Fix M "generic enough" points (p_i, q_i) on the parabola $q = p^2$, with

$$p_1 < p_2 < \cdots < p_M.$$

- For $I = \{i_1 < \cdots < i_N\} \in \binom{[M]}{N}$, let

$$\Theta_I = \sum_{i \in I} p_i x + q_i y + \omega_i(\mathbf{t}).$$

- Here $\mathbf{t}$ is the *multi-time parameter* $(t_3, \ldots, t_M)$ and

$$\omega_i(\mathbf{t}) = \sum_{k=3}^{M-1} p_i^k t_k.$$

From matrices to soliton solutions

- Let $K_I = \prod_{\ell<m}(p_{i_m} - p_{i_\ell})$
- Let A be a matrix representing a point in $\mathrm{Gr}_{\geq 0}(N, M)$.
- $$A \rightsquigarrow u_A(x, y, \mathbf{t}) = 2\frac{\partial^2}{\partial x^2}(\ln \tau_A(x, y, \mathbf{t}))$$

  where we have

  $$\tau_A = \sum_{I \in \mathcal{M}(A)} \Delta_I(A) K_I \exp(\Theta_I)$$

Contour plots

- The function $u_A(x, y, t)$ models the height of a wave at time t.
- Wave peaks give a *contour plot*.

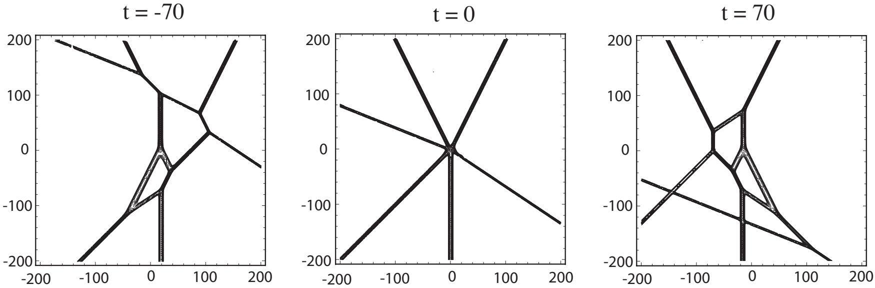


Figure: Contour plots corresponding to a point in $Gr_{\geq 0}(3, 6)$

Contour plots are tropical curves

- Can approximate the contour plot as the locus where

$$f_A = \max\{\ln(\Delta_I(A)K_I) + \Theta_I : I \in \mathcal{M}(A)\}$$

is nonlinear.

- So f_A chops up the (x, y)-plane into regions where one plane

$$z = \ln(\Delta_I(A)K_I) + \Theta_I(x, y, \mathbf{t})$$

is dominant over the others.

Asymptotic contour plots

- Asymptotic contour plots: rescale the variables, can assume the scalars $\Delta_I(A)$ are negligible.
- The *asymptotic contour plot* for fixed multi-time parameter $\mathbf{t_0}$ is the locus where
$$\hat{f}_{\mathcal{M}} = \max\{\Theta_I(x, y, \mathbf{t_0}) \mid I \in \mathcal{M}(A)\}$$
is non-linear.
  - More precisely, $\hat{f}_{\mathcal{M}} = \lim_{s\to\infty} \frac{1}{s} f_A(sx, sy, s\mathbf{t}_0)$.

Example

- Let $N = 1$, $M = 4$.
- Choose parameters

$$p_1 = -2 \qquad p_2 = 0 \qquad p_3 = 1 \qquad p_4 = 2.$$

- Let $t = t_3 = 1$.
- Then we have

$$\begin{aligned}
\theta_1(x, y, t) &= -2x + 4y - 8 \\
\theta_2(x, y, t) &= 0 \\
\theta_3(x, y, t) &= x + y + 1 \\
\theta_4(x, y, t) &= 2x + 4y + 8
\end{aligned}$$

Example Continued

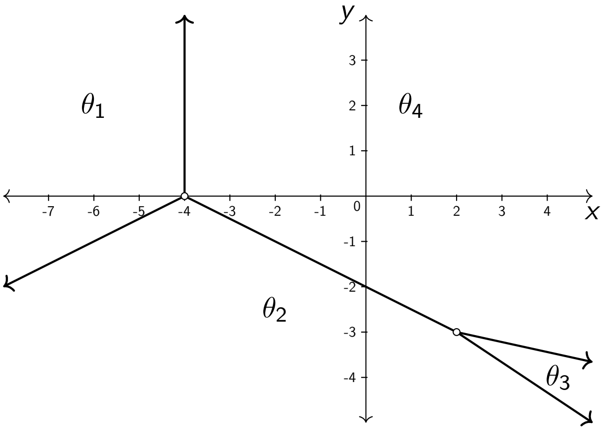


Figure: An asymptotic contour plot.

Soliton graphs

- Goal: understand combinatorics of asymptotic contour plots.
- Take plots up to isotopy, get *soliton graphs*.
- Restrict to $\mathrm{Gr}_{>0}(N, M)$, where all Plücker coordinates positive.
  - In this case, soliton graphs are *plabic graphs* (**?**).
- Want to classify soliton graphs for $\mathrm{Gr}_{>0}(N, M)$.

Constructing soliton graphs

- Embed asymptotic contour plot in a disk, take graph up to isotopy.
- Color an internal vertex white if the adjoining regions share $M - 1$ indices, and black otherwise.

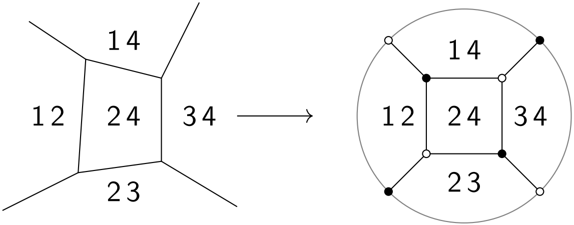
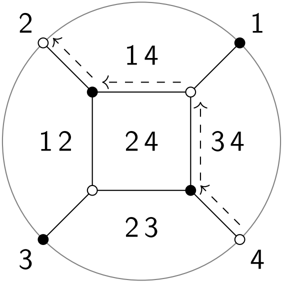


Figure: From contour plot to soliton graph.

Plabic graphs

- Planar, bicolored graph embedded in a disk, satisfies some technical conditions.
- We label each face F plabic graph with an i if F is to the left of the *zig-zag path* T_i ending at boundary vertex i.



Face labels of plabic graphs

- The face labels determine the graph, up to contracting and un-contracting unicolored edges.
- Face labels of plabic graphs give *clusters* in the cluster algebra structure of $\mathrm{Gr}(N, M)$ (**?**).
    - Every cluster containing only Plücker variables comes from a plabic graph.

Weak Separation

- A collection $\mathcal{C}$ of N-element subsets of $\{1, 2, \ldots, M\}$ is the set of face labels of a plabic graph for $\mathrm{Gr}(N, M)$ if and only if it is a *maximal weakly separated collection*.
  - For any $I, J \in \mathcal{C}$, if the numbers $1, \ldots, M$ are arranged in order around a circle, we can draw a chord that separates $I \backslash J$ from $J \backslash I$.
  - $\mathcal{C}$ has $N(M - N) + 1$ elements (so it is as large as possible).

Realizability

Theorem (?)

Every soliton graph for $\mathrm{Gr}_{>0}(N, M)$ *is a plabic graph. Face labels of the plabic graph correspond to dominant exponentials of the soliton solution.*

- We say a collection of face labels is *realizable* if it comes from a soliton graph.
- Goal: classify realizable collections.

The duality map

- Map a plane to a point:

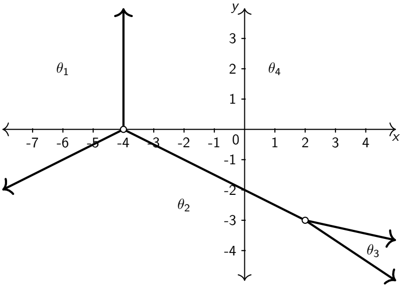
$$\theta_i(x,y) = p_i x + q_i y + \omega_i \qquad \mapsto \qquad \hat{\mathbf{v}}_i = (p_i, q_i, \omega_i)$$

- Take convex hull of points, project from above to (p,q)-plane.
- Get triangulation of the M-gon.

Example continued

- Recall:

$$p_1 = -2 \qquad p_2 = 0 \qquad p_3 = 1 \qquad p_4 = 2 \qquad t = 1$$



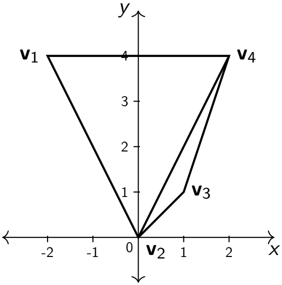
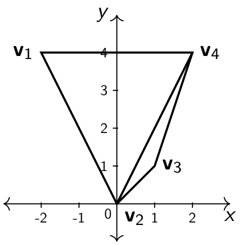
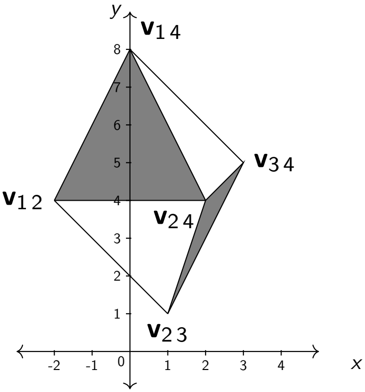


Figure: A soliton tiling.

The $N = 2$ case

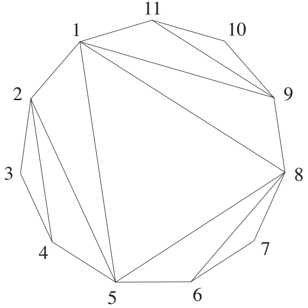
- $$\Theta_I(x,y) \qquad \mapsto \qquad \hat{\mathbf{v}}_I = \sum_{i \in I} \hat{\mathbf{v}}_i$$





Induction

- Use *induction algorithm* to construct tiling for $\mathrm{Gr}_{>0}(N+1, M)$ from tiling for $\mathrm{Gr}_{>0}(N, M)$ (**?**).



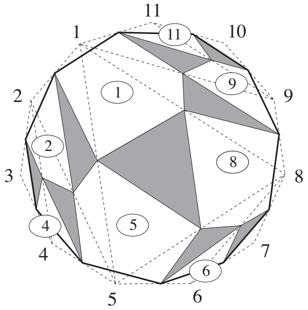


Figure: Using the induction algorithm.

Induction continued

- Triangulation of the white polygons depends on the weights of our points.

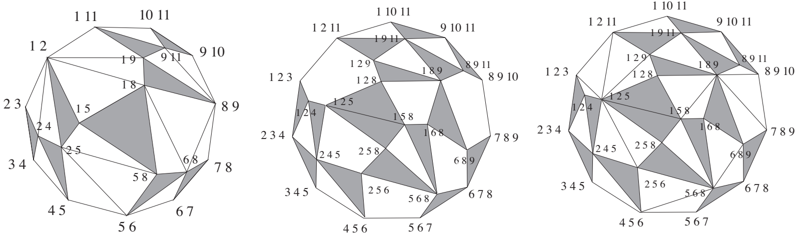


Figure: From $N = 2$ to $N = 3$.

Summary of results

- Question: is every maximal weakly separated collection realizable?
- Answer is yes for...
  - $\mathrm{Gr}_{>0}(2, M)$ (**?**).
  - $\mathrm{Gr}_{>0}(3, 6)$ (**?**).
  - $\mathrm{Gr}_{>0}(3, 7), \mathrm{Gr}_{>0}(3, 8)$ [Kodama and K.]
- In general, the answer is no.

Choices of parameters

- For $N = 3$, $M = 6, 7$ or 8, every weakly separated collection is realizable for *some* choice of parameters $p_1, p_2, \ldots, p_M$.
- Which plabic graphs we can realize depends on our choice of p_i.

Examples

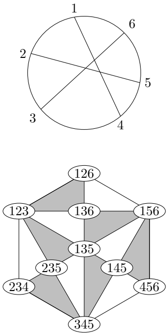


Figure: Triangulations which are only realizable for some choices of parameters.

Classification for $\mathrm{Gr}(3,6)$ and $\mathrm{Gr}(3,7)$.

- For $\mathrm{Gr}(3,6)$, there are 34 possible graphs, each generic choice of parameters lets us realize 32 of them (**?**).
- For $\mathrm{Gr}(3,7)$, there are 259 possible graphs, a each generic choice of p_i we can realize 231 of them [Kodama and K].
  - Main obstacle: same as in $\mathrm{Gr}(3,6)$ case.
- For $\mathrm{Gr}(3,8)$, don't yet have classification.

The general case

- Not all weakly separated collections are realizable.
- Can build plabic graph from any simple, non-stretchable arrangement of pseudolines, which gives a counter-example [Thomas, 2017].
  - Smallest counter-example of this form is for $\mathrm{Gr}(9, 18)$
  - Conjecture: much smaller counter-examples exist.

References I