Approaching quasi-period collapse via orbifold singularities

Ben Wormleighton

UC Berkeley

Main references

- *B. Wormleighton; Reconstructing singularities on orbifold del Pezzo surfaces from their Hilbert series (2018)*
- *A. M. Kasprzyk, B. Wormleighton; Quasi-period collapse for duals to Fano polygons (2018)*

Disclaimer

This is a talk intended for both combinatorists and geometers. I will *seek* to appease both audiences.

Setting - Combinatorics

- Consider a polytope $P \subset \mathbb{R}^d$; that is, the convex hull of a finite set of points in $\mathbb{R}^d$.
- I will assume that P is full-dimensional (i.e. a d-dimensional subset of $\mathbb{R}^d$)
- A basic problem is to count the lattice points in dilates of P: define the *Ehrhart function* of P to be

$$L_P(n) := \# nP \cap \mathbb{Z}^d$$

Theorem (Ehrhart, '62)

Suppose P is a **lattice** *polytope of dimension d. $L_P(n)$ is given by a polynomial of degree d called the* Ehrhart polynomial *of P.*

- Some coefficients are known explicitly:

$$\mathrm{coeff}_1 = 1, \mathrm{coeff}_{n^d} = \mathrm{Vol}(P), \mathrm{coeff}_{n^{d-1}} = \frac{1}{2}\mathrm{Vol}(\partial P)$$

- In two dimensions this describes $L_P(n)$ completely

$$L_P(n) = \mathrm{Vol}(P)n^2 + \frac{1}{2}\#\partial P \cap \mathbb{Z}^d \cdot n + 1$$

and at $n = 1$ recovers Pick's formula

$$\mathrm{Vol}(P) = \#\,\mathrm{Int}(P) \cap \mathbb{Z}^2 + \frac{1}{2}\#\partial P \cap \mathbb{Z}^2 - 1$$

Rational Ehrhart theory

- A *rational polytope* is a polytope whose vertices lie in $\mathbb{Q}^d$.

Theorem (Ehrhart,'62)

Suppose P is a rational polytope of dimension d. $L_p(n)$ is given by a quasi-polynomial *of degree d: that is, there are degree d polynomials $L_0(n), \ldots, L_{\pi-1}(n)$ such that*

$$L_P(n) = L_i(n) \text{ when } n \equiv i \bmod \pi$$

This quasi-polynomial is called the Ehrhart quasi-polynomial of P.

- The smallest value of π is called the *quasi-period* of P (or of L_P) and denoted π_P.
- It divides all other π for which such $L_0, \ldots, L_{\pi-1}$ exist.

Example

Consider the triangle P with vertices $(0,0), (\frac{1}{2},0), (0,\frac{1}{2})$. This has

$$L_P(n) = \begin{cases} \frac{1}{8}(n+2)(n+4) & n \equiv 0 \bmod 2 \\ \frac{1}{8}(n+1)(n+3) & n \equiv 1 \bmod 2 \end{cases}$$

Quasi-period collapse

- Define the *denominator* r_P of P by

$$r_P := \mathrm{lcm}\{\text{denominators of vertices of } P\}$$

- Note that $r_P P$ is a lattice polytope and hence $L_{r_P P}(n) = L_P(r_P n)$ is given by a polynomial.

Lemma

π_P *divides* r_P.

- However, π_P is **not** equal to r_P in general.

Example

Let P be the triangle with vertices $(1,0),(0,1),(-\frac{1}{2},-\frac{1}{2})$. This has denominator 2 but

$$L_P(n) = n^2 + n + 1$$

and so $\pi_P = 1$.

Example

Consider the triangle P with vertices $(0,0),(0,6),(\frac{3}{2},0)$. This also has denominator 2 but

$$L_P(n) = \frac{9}{2}n^2 + \frac{9}{2}n + 1$$

and so $\pi_P = 1$.

Definition

A rational polytope P undergoes *quasi-period collapse* if $\pi_P < r_P$.

What's known?

- De Loera–McAllister '03: First infinite family of rational polytopes undergoing quasi-period collapse, occurring in Gelfand-Tsetlin polytopes
- McAllister–Woods '05: Provide an abstract characterisation of rational polygons of quasi-period 1 (roughly that they obey Pick's theorem).
- Beck–Sam–Woods '07: Compute the periods of coefficients in terms of regularity of associated generating functions and get precise results for the second leading coefficient.

What's known?

- Haase–McAllister '07: Two rational polytopes P, Q have the same Ehrhart function iff they are $\mathrm{GL}_n(\mathbb{Z})$-*scissors equivalent*. One can find rational polytopes equivalent to lattice polytopes - hence of quasi-period 1 - using this method.
- Cristofaro-Gardiner–Li–Stanley '15: Study irrational simplices with quasi-period 1 arising from symplectic geometry.
- McAllister–Moriarity '15: Construct rational polygons whose coefficients have periods $(1, r, s)$ for any $r, s \in \mathbb{Z}_{\geq 1}$.

What's known?

In summary...

What's known?

In summary... not much.

Central questions

- Are there other (potentially illuminating) characterisations of quasi-period collapse?
- Can one systematically produce interesting examples of quasi-period collapse?

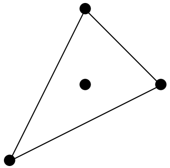
Setting - Geometry

Toric geometry

- For a lattice polytope $P \subset \mathbb{R}^d$ one can associate a *toric variety* X_P: this is a compactification of an algebraic torus $(\mathbb{C}^\times)^d$ by torus orbits.

Example

I use the ‘face fan’ convention and so this is a polytope for $\mathbb{P}^2$:



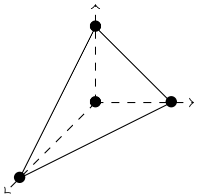
Setting - Geometry

Toric geometry

- For a lattice polytope $\Delta \subset \mathbb{R}^d$ one can associate a *toric variety* X_Δ: this is a compactification of an algebraic torus $(\mathbb{C}^\times)^d$ by torus orbits.

Example

I use the 'face fan' convention and so this is a polytope for $\mathbb{P}^2$:



- This toric variety X_Δ comes with an ample divisor D_Δ. A basis for sections of the line bundle $\mathcal{O}(D_\Delta)$ is in bijection with lattice points in the *dual polytope*

$$\Delta^\vee := \{v \in (\mathbb{R}^d)^\vee : \langle u, v\rangle \geq -1 \text{ for all } u \in \Delta\}$$

- Taking tensor powers of line bundles corresponds to dilating polytopes and so one has an equality

$$h^0(nD_\Delta) = L_{\Delta^\vee}(n)$$

- $\Delta^\vee$ is **not** usually a lattice polytope even if Δ is.

Fano varieties / polytopes

- The *canonical divisor* of a smooth variety is $K_X := \wedge^{\text{top}} T_X^*$. One can make a similar construction for singular X.
- A variety is *Fano* if $-K_X$ is ample. Note that I allow $-K_X$ to be $\mathbb{Q}$-Cartier.
- A polytope is *Fano* if
  - its vertices are primitive lattice points
  - it contains the origin in its interior

Theorem

Toric Fano varieties of dim $d \longleftrightarrow$ Fano polytopes of dim d.

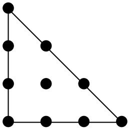
- The bijection takes a Fano polytope Δ to the toric Fano variety X_Δ, for which the ample divisor D_Δ is $-K_{X_\Delta}$.
- Hence $h^0(-nK_{X_\Delta}) = L_{\Delta^\vee}(n)$.
- One can add adjectives on either side of the bijection:
  - smooth $\leftrightarrow$ smooth
  - Gorenstein $\leftrightarrow$ reflexive
  - orbifold $\leftrightarrow$ simplicial

Example

Continuing with $X = \mathbb{P}^2$, one has $-K_X = \mathcal{O}(3)$ and so

$$h^0(-nK_X) = \binom{2+3n}{2} = \frac{9}{2}n^2 + \frac{9}{2}n + 1$$

The dual polytope for $\mathbb{P}^2$ is



that indeed has the right Ehrhart polynomial.

- The *Gorenstein index* r_X of a Fano variety X is the smallest positive integer r such that rK_X is Cartier.

Lemma

$r_{X_\Delta} = r_{\Delta^\vee}$

- There is thus a natural geometric interpretation of the denominator of a polytope.
- If we can also find a geometric interpretation of the quasi-period then perhaps we can study quasi-period collapse via toric geometry...

Main results

- It will turn out to be more fruitful to consider power series than polynomials.
- Define the *Hilbert series* of (X, D) and the *Ehrhart series* of P respectively to be

$$\mathrm{Hilb}_{(X,D)}(t) := \sum_{n\geq 0} h^0(nD)t^n \text{ and } \mathrm{Ehr}_P(t) := \sum_{n\geq 0} L_P(n)t^n$$

Structural assumptions

- Assume that X is a *orbifold del Pezzo surface*: a Fano variety of dimension 2 with only *orbifold singularities*.
- These are isolated singularities that locally look like

$$\mathbb{C}^2/G$$

  where $G \subset \mathrm{GL}_2(\mathbb{C})$ is a finite cyclic subgroup. Write $\frac{1}{r}(a,b)$ for the singularity where G is generated by $\begin{pmatrix} \varepsilon^a & \\ & \varepsilon^b \end{pmatrix}$ with ε a primitive rth root of unity.
- Every polygon is simplicial and so every toric del Pezzo surface is an orbifold del Pezzo surface.

- For such spaces we have the following:

Theorem (Reid '87 + Akhtar-Kasprzyk '13)

For X an orbifold del Pezzo surface with singular locus $\mathcal{B}_X$ one has

$$\mathrm{Hilb}_{(X,-K_X)}(t) = \frac{1 + (K_X^2 - 2)t + t^2}{(1-t)^3} + \sum_{\sigma \in \mathcal{B}_X} Q_\sigma$$

where, if σ is a singularity of type $\frac{1}{r}(a,b)$, the orbifold contribution

$$Q_\sigma = \frac{1}{1-t^r} \sum_{i=0}^{r} (\delta_{r,a,b,i} - \delta_{r,a,b,0}) t^{i-1}$$

- The $\delta_{r,a,b,i}$ are `strange` quantities called Dedekind sums.

- In particular, as expected, the Hilbert function of a toric del Pezzo surface X_Δ is given by a quasipolynomial.
- We know that this is equal to $L_{\Delta^\vee}$ and so we can use the decomposition result to study quasi-period collapse for $\Delta^\vee$.
- Denote by π_X the quasi-period of $h^0(-nK_X)$.

- Define the *local index* ℓ_σ of an orbifold singularity σ of type $\frac{1}{r}(a,b)$ to be $r/\gcd(r,a+b)$.
- The important takeaway from the formula is...

Lemma (W. '18)

$Q_\sigma = \sum_{n\geq 0} q(n)t^n$ *where* $q(n)$ *is a quasipolynomial of quasi-period* ℓ_σ*.*

- The precise way the singularities on X contribute to the periodicity of the Hilbert function is controlled by the local index.

Combinatorial interlude

- Singularities on X_Δ correspond to faces of Δ.
- The local index of σ is the 'lattice height' of the face away from the origin.
- This will correspond to a vertex in the dual with height $1/\ell_\sigma$.

- We have

$$\mathrm{Hilb}_{(X,-K_X)}(t) = \text{polynomial term} + \sum_{\sigma \in \mathcal{B}_X} \text{quasipolynomial terms}$$

Lemma (Akhtar-Kasprzyk '13 + W. '18)

If X is an orbifold del Pezzo surface then $r_X = \mathrm{lcm}\{\ell_\sigma : \sigma \in \mathcal{B}_X\}$.

- The disparity between r_X and π_X is then measured by 'how much cancellation' happens between the Q_σ.

Cancelling tuples

Definition

A collection $\{\sigma_1, \ldots, \sigma_r\}$ of orbifold singularities is a *cancelling tuple* if

$$\sum_{i=1}^{r} Q_{\sigma_i} = 0$$

- These encode relations between the Q_σ.

Definition

Let X be an orbifold del Pezzo surface with singular locus $\mathcal{B}_X$. A subset $\mathcal{S} \subset \mathcal{B}_X$ is *invisible* if

$$\sum_{\sigma \in \mathcal{S}} Q_\sigma = 0$$

An *invisible basket* for X is a maximal invisible subset of $\mathcal{B}_X$.

- These are not unique in general.

Theorem (Kasprzyk-W. '18)

Let X be an orbifold del Pezzo surface with singular locus $\mathcal{B}_X$. Let $\mathcal{IB}_X$ be an invisible basket for X. Then

$$\pi_X = \operatorname{lcm}\{\ell_\sigma : \sigma \in \mathcal{B}_X \setminus \mathcal{IB}_X\}$$

Corollary

For such spaces,

$$\frac{r_X}{\pi_X} = \frac{\operatorname{lcm}\{\ell_\sigma : \sigma \in \mathcal{IB}_X\}}{\gcd\{\operatorname{lcm}\{\ell_\sigma : \sigma \in \mathcal{B}_X \setminus \mathcal{IB}_X\}, \operatorname{lcm}\{\ell_\sigma : \sigma \in \mathcal{IB}_X\}\}}$$

Translation into Ehrhart theory

- Say that a polygon is *dual-Fano* if it is dual to a Fano polygon.
- Define the *singular locus* $\mathcal{B}_P$ of a dual-Fano polygon P to be the singular locus of the corresponding toric del Pezzo surface $X_{P^\vee}$. This can be read directly from the polygon.
- An *invisible basket* for P is an invisible basket for $X_{P^\vee}$.

Theorem (Kasprzyk-W. '18)

Let P be a dual-Fano polygon with singular locus $\mathcal{B}_P$. Let $\mathcal{IB}_P$ be an invisible basket for P. Then

$$\pi_P = \operatorname{lcm}\{\ell_\sigma : \sigma \in \mathcal{B}_P \setminus \mathcal{IB}_P\}$$

Corollary

For such polygons,

$$\frac{r_P}{\pi_P} = \frac{\operatorname{lcm}\{\ell_\sigma : \sigma \in \mathcal{IB}_P\}}{\gcd\{\operatorname{lcm}\{\ell_\sigma : \sigma \in \mathcal{B}_P \setminus \mathcal{IB}_P\}, \operatorname{lcm}\{\ell_\sigma : \sigma \in \mathcal{IB}_P\}\}}$$

Example

Let P be the triangle with vertices $(1,0),(0,1),(-\frac{1}{2},-\frac{1}{2})$ from a previous example. This has $r_P = 2$ and $\pi_P = 1$.
$P^\vee$ is a polygon for the weighted projective plane $\mathbb{P}(1,1,2)$, which has $\mathcal{B} = \{\frac{1}{2}(1,1)\}$ and

$$Q_{\frac{1}{2}(1,1)} = 0$$

so that $\mathcal{IB}_P = \mathcal{B} = \{\frac{1}{2}(1,1)\}$.
It follows that $\pi_P = 1$ and $r_P = 2$.

Geometric insight

- This presentation of the denominator and quasi-period allows us to apply geometric tools to study quasi-period collapse.
- We primarily use deformation theory:

Lemma

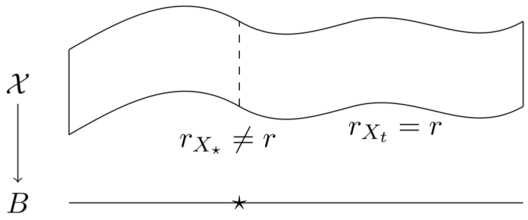
If two orbifold del Pezzo surfaces are $\mathbb{Q}$-Gorenstein (qG) deformation-equivalent then their Hilbert series are equal.

Corollary

If σ is a qG-smoothable singularity then $Q_\sigma = 0$.

Big picture

Let $\mathcal{X} \to B$ be a qG family of toric del Pezzo surfaces. The Hilbert function (and its quasi-period) is constant across fibres but the Gorenstein index (or denominator) can change.



Quasi-period collapse arises when special fibres gain some new singularities of different local index.

Degenerations of $\mathbb{P}^2$

Theorem (Hacking-Prokhorov '10)

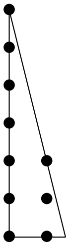
The qG-degenerations of $\mathbb{P}^2$ *are weighted projective planes* $\mathbb{P}(a^2, b^2, c^2)$ *where* (a, b, c) *is a* Markov triple *satisfying*

$$a^2 + b^2 + c^2 = 3abc$$

- The polygons for weighted projective planes are triangles whose vertices span $\mathbb{Z}^2$.
- $X = \mathbb{P}(a^2, b^2, c^2)$ gives a rational triangle with arbitrarily large denominator $r_X = abc$ and quasi-period $\pi_X = \pi_{\mathbb{P}^2} = 1$.

Example

$(1, 1, 2)$ is a Markov triple giving the weighted projective plane $\mathbb{P}(1, 1, 4)$. The corresponding dual-Fano polygon is the triangle with vertices $(0, 0), (0, 6), (\frac{3}{2}, 0)$ from before with denominator $2 = abc$ and quasi-period 1.



Smoothable singularities

Theorem (Kollár-Shepherd-Barron '88)

The qG-smoothable orbifold singularities (or T-singularities) are those of the form $\frac{1}{dn^2}(1, dnc - 1)$.

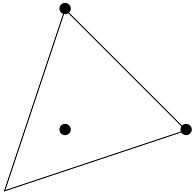
- If τ is a T-singularity as above, $\ell_\tau = n$ and $\mathrm{width}(\tau) = dn$.
- These are the singularities whose faces have width a multiple of their height.

Example

$\frac{1}{2}(1, 1)$ is a T-singularity of local index 1 and width 2. Singularities of local index 1 are Du Val singularities (and are qG-smoothable).

Example

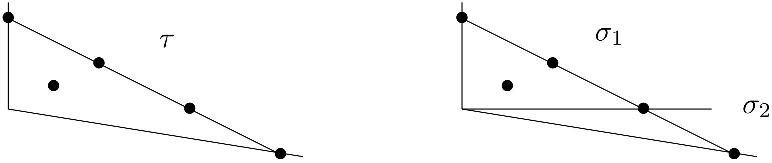
$\mathbb{P}(1,1,2)$ has only a $\frac{1}{2}(1,1)$ singularity and has as its dual-Fano polygon the rational triangle below seen previously, which has denominator 2 and quasi-period 1.



Shattering

Example

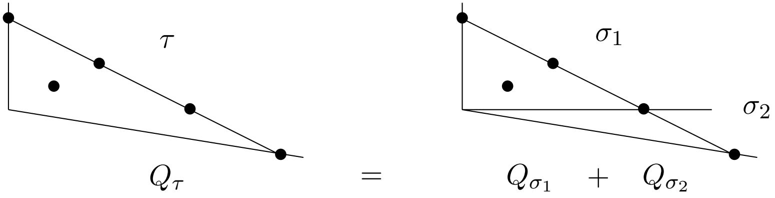
Let τ be the T-singularity $\frac{1}{9}(1,2)$. Inserting a ray through the third lattice point along the face as shown divides τ into two singularities $\sigma_1 = \frac{1}{6}(1,1)$ and $\sigma_2 = \frac{1}{3}(1,1)$.



- In geometry, inserting a ray like this corresponds to a *crepant blowup* $\varphi : \widetilde{X} \to X$.
- Crepant means that $\varphi^* K_X = K_{\widetilde{X}}$ and so the Hilbert functions of $(X, -K_X)$ and $(\widetilde{X}, -K_{\widetilde{X}})$ agree.
- In particular, the orbifold contributions for singularities on X and $\widetilde{X}$ agree.

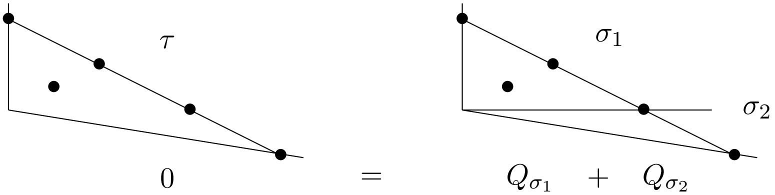
Example

In the previous example, this means that:



Example

Since τ is smoothable:



and so $\{\frac{1}{3}(1,1), \frac{1}{6}(1,1)\}$ is a cancelling tuple.

- We say that $\{\sigma_1, \sigma_2\}$ is a *shattering* of τ. Any collection of singularities $\{\sigma_1, \ldots, \sigma_n\}$ obtained in this way from successive crepant blowups of a smoothable singularity will have

$$\sum_{i=1}^{n} Q_{\sigma_i} = 0$$

Example (Real-life example)

The quotient $\mathbb{P}(1, 2, 3)/(\mathbb{Z}/3)$ has a $\frac{1}{3}(1, 1)$ and a $\frac{1}{6}(1, 1)$ singularity.

Cancelling tuple conjecture

Every cancelling tuple comes from shattering a T-singularity.

- There is much computational evidence for the conjecture (a version of it has been verified up to local index 34) and some partial results.
- It is equivalent to the following conjecture. Let Q_σ^{num} be the numerator of Q_σ in lowest terms.

Δ-lattice conjecture

The rank of the lattice in $\mathbb{Q}[t]_{\ell-1}$ spanned by Q_σ^{num} as σ ranges over orbifold singularities of local index ℓ is $\frac{1}{2}\varphi(\ell)+1$

Summary

Key insights

- the local index controls how singularities contribute to the quasi-period
- quasi-period collapse is measured by cancellations among the singular locus
- the qG-deformation theory of orbifold del Pezzo surfaces gives many (likely all) examples of cancellation

Future directions

- Independent of the status of the conjectures above, the approach from singularity / deformation theory suggests an approach to new problems pertaining to quasi-period collapse.

Problem

Classify all (dual-Fano) rational triangles of quasi-period 1.

Example

- The degenerations of $\mathbb{P}^2$ give all the smoothable examples.
- $\mathbb{P}(1,2,3)/(\mathbb{Z}/3)$ has a cancelling pair $\{\frac{1}{3}(1,1), \frac{1}{6}(1,1)\}$ and a T-singularity $\frac{1}{9}(1,8)$ and so has quasi-period 1.

(Dual-)Fano triangles correspond to *fake weighted projective planes*: quotients of weighted projective planes by finite cyclic groups.

Problem, v.2

- *Classify all fake weighted projective planes with a cancelling pair and a T-singularity.*
- *Classify all fake weighted projective planes with a cancelling triple.*

This is now a ‘numerical’ problem.

Other problems

- Pushing past Fano + dimension 2 assumptions
- Resolving the cancelling tuple / Δ-lattice conjecture
- Studying 'genericity' of quasi-period collapse

References

- *Singularity content*, M. E. Akhtar & A. M. Kasprzyk ('14)
- *Quasi-period collapse and* $\mathrm{GL}_n(\mathbb{Z})$*-scissors congruence in rational polytopes*, C. Haase & T. B. McAllister ('08)
- *Smoothable del Pezzo surfaces with quotient singularities*, P. Hacking & Y. Prokhorov ('10)
- *Quasi-period collapse for duals to Fano polygons*, A. M. Kasprzyk & B. Wormleighton ('18)
- *Threefolds and deformations of surface singularities*, J. Kollár & N. Shepherd-Barron ('88)
- *Young person's guide to canonical singularities*, M. Reid ('87)
- *Reconstruction of singularities on orbifold del Pezzo surfaces from their Hilbert series*, B. Wormleighton ('18)