VI. Appendix

Proposition 1. *For any network (X, ω_X), the minimum chain cost $u_X^{\mathcal{H}}$ satisfies the strong triangle inequality.*

Proof. Let $x, x', x'' \in X$. We wish to show:

$$u_X^{\mathcal{H}}(x, x'') \leq \max\left(u_X^{\mathcal{H}}(x, x'), u_X^{\mathcal{H}}(x', x'')\right).$$

Let c', c'' be chains such that $u_X^{\mathcal{H}}(x, x') = \mathrm{cost}_X(c')$ and $u_X^{\mathcal{H}}(x', x'') = \mathrm{cost}_X(c'')$. Consider the composed chain $c := c'' \circ c'$ beginning at x and ending at x''. Then $\mathrm{cost}_X(c) \leq \max\left(\mathrm{cost}_X(c'), \mathrm{cost}_X(c'')\right)$. The result follows. □

Theorem 2. *Let X be a finite set and let ω_1 and ω_2 be two different weight functions defined on $X \times X$. Write $(X, u_1^{\mathcal{H}}) := \mathcal{H}(X, \omega_1)$ and $(X, u_2^{\mathcal{H}}) := \mathcal{H}(X, \omega_2)$. Then we have:*

$$\|u_1^{\mathcal{H}} - u_2^{\mathcal{H}}\|_{\ell^\infty(X \times X)} \leq \|\omega_1 - \omega_2\|_{\ell^\infty(X \times X)}.$$

Proof. Let $x, x' \in X$ be such that $\|u_1^{\mathcal{H}} - u_2^{\mathcal{H}}\|_{\ell^\infty(X \times X)} = |u_1^{\mathcal{H}}(x, x') - u_2^{\mathcal{H}}(x, x')|$. Then we have

$$|\overline{\omega}_1(x, x') - \overline{\omega}_2(x, x')| \leq \|\omega_1 - \omega_2\|_{\ell^\infty(X \times X)}.$$

Next let $c \in C_{(X,\omega_1)}(x, x')$ be an optimal chain, i.e. a chain such that

$$\max_{x_i, x_{i+1} \in c} \overline{\omega}_1(x_i, x_{i+1}) = u_1^{\mathcal{H}}(x, x').$$

Here we are writing $c = \{x_0 = x, x_1, \ldots, x_n = x'\}$. Then for any $1 \leq i \leq n$, we obtain:

$$\begin{aligned}\overline{\omega}_2(x_i, x_{i+1}) &\leq \overline{\omega}_1(x_i, x_{i+1}) + \|\omega_1 - \omega_2\|_{\ell^\infty(X \times X)} \\ &\leq u_1^{\mathcal{H}}(x, x') + \|\omega_1 - \omega_2\|_{\ell^\infty(X \times X)}.\end{aligned}$$

Since this holds for any $i \in \{1, \ldots, n\}$, we can minimize over all chains to obtain:

$$u_2^{\mathcal{H}}(x, x') - u_1^{\mathcal{H}}(x, x') \leq \|\omega_1 - \omega_2\|_{\ell^\infty(X \times X)}.$$

A similar argument shows:

$$u_1^{\mathcal{H}}(x, x') - u_2^{\mathcal{H}}(x, x') \leq \|\omega_1 - \omega_2\|_{\ell^\infty(X \times X)}.$$

This concludes the proof. □

Proposition 3 (Property A1)**.** *For the two-point network $(X, \omega_X) = (\{p, q\}, \left(\begin{smallmatrix}\alpha & \gamma \\ \delta & \beta\end{smallmatrix}\right))$, we have $\mathcal{H}(X, \omega_X) = (\{p, q\}, \left(\begin{smallmatrix}\alpha & \Gamma \\ \Delta & \beta\end{smallmatrix}\right))$, where $\Gamma = \max\{\alpha, \beta, \gamma\}$ and $\Delta = \max\{\alpha, \beta, \delta\}$.*

Proof. Note that the minimum cost chain from p to itself is the stationary chain, so $u_X^{\mathcal{H}}(p, p) = \alpha$. Similarly $u_X^{\mathcal{H}}(q, q) = \beta$. The values Γ and Δ appear from the definition of $\overline{\omega}_X$. □

Proposition 4 (Property A2)**.** *If $\phi : X \to Y$ satisfies $\omega_X(x, x') \geq \omega_Y(\phi(x), \phi(x'))$ for all $x, x' \in X$, then we also have $u_X^{\mathcal{H}}(x, x') \geq u_Y^{\mathcal{H}}(\phi(x), \phi(x'))$ for all $x, x' \in X$.*

Proof. It follows from the assumption that we have:

$$\overline{\omega}_X(x, x') \geq \overline{\omega}_Y(\phi(x), \phi(x')) \text{ for all } x, x' \in X.$$

Let $c \in C_X(x, x')$ be a chain such that $\mathrm{cost}_X(c) = u_X^{\mathcal{H}}(x, x')$. Write $c = \{x = x_0, \ldots, x_n = x'\}$. Consider the chain $\phi(c) = \{\phi(x_0), \ldots, \phi(x_n)\}$. Then $\phi(c) \in C_Y(\phi(x), \phi(x'))$, and $\mathrm{cost}_Y(\phi(c)) \leq \mathrm{cost}_X(c)$ by the preceding observation. The result follows. □

Theorem 5. *For any n-point space $X \in \mathcal{N}$, write $(X, u_X) = \mathcal{H}(X, \omega_X)$. Then we have:*

$$\|\omega_X - u_X\|_{l^\infty(X \times X)} \leq \log_2(2n)\, \mathrm{ult}(X).$$

Moreover, this bound is asymptotically tight.

Proof. Let $\delta = \mathrm{ult}(X, d_X)$. First we claim that for any sequence of $2^k + 1$ points, we have:

$$\max_{1 \leq i \leq 2^k} \overline{\omega}_X(x_i, x_{i+1}) \geq \overline{\omega}_X(x_1, x_{2^k+1}) - k\delta.$$

To see this, we proceed by induction. Notice that for $k = 1$, we have the following by the definition of the ultranetwork constant:

$$\begin{aligned}\omega_X(x_1, x_3) &\leq \max\left(\omega_X(x_1, x_2), \omega_X(x_2, x_3)\right) + \delta \\ &\leq \max\left(\overline{\omega}_X(x_1, x_2), \overline{\omega}_X(x_2, x_3)\right) + \delta.\end{aligned}$$

Also, we have

$$\begin{aligned}&\omega_X(x_1, x_1) \leq \overline{\omega}_X(x_1, x_2), \\ &\omega_X(x_3, x_3) \leq \overline{\omega}_X(x_2, x_3).\end{aligned}$$

Then it follows that:

$$\overline{\omega}_X(x_1, x_3) \leq \max\left(\overline{\omega}_X(x_1, x_2), \overline{\omega}_X(x_2, x_3)\right) + \delta.$$

This proves the base case. Next let $k \in \mathbb{N}$, and suppose the claim holds for k. By the base case, we obtain:

$$\overline{\omega}_X(x_1, x_{2^{k+1}+1}) \leq \max\left(\overline{\omega}_X(x_1, x_{2^k+1}), \overline{\omega}_X(x_{2^k+1}, x_{2^k+1+2^k})\right) + \delta.$$

But by the induction step, we have:

$$\begin{aligned}\overline{\omega}_X(x_1, x_{2^k+1}) &\leq \max_{1 \leq i \leq 2^k} \overline{\omega}_X(x_i, x_{i+1}) + k\delta \\ \overline{\omega}_X(x_{2^k+1}, x_{2^k+1+2^k}) &\leq \max_{2^k+1 \leq i \leq 2^{k+1}} \overline{\omega}_X(x_i, x_{i+1}) + k\delta\end{aligned}$$

Thus, taking the maximum of the two, we obtain:

$$\overline{\omega}_X(x_1, x_{2^{k+1}+1}) \leq \max_{1 \leq i \leq 2^{k+1}} \overline{\omega}_X(x_i, x_{i+1}) + (k+1)\delta.$$

This proves the claim. Next, let $x, x' \in X$. Let $c \in C(x, x')$. Write $c = \{x = x_1, \ldots, x_p = x'\}$. Note that if c contains any repetition, i.e. if there exist $i < j \leq p$ with $x_i = x_j$, then we may replace c by $c' = \{x_1, \ldots, x_i, x_{j+1}, \ldots, x_p\}$. Thus by reindexing if necessary, we obtain a chain of distinct elements $c' = \left\{x = x'_1, \ldots, x'_q = x'\right\}$, with $q < p$. Also note that $\mathrm{cost}(c') \leq \mathrm{cost}(c)$. Next let k be the greatest integer such that $2^k \leq n$. Then we have $n \leq 2^{k+1} \leq 2n$. Since c' has length $q \leq n$, we can define:

$$\bar{c} = \left\{x'_1, \ldots, x'_q, x'_q, \ldots, x'_q\right\},$$

where $\bar{c}$ is obtained from c' by padding copies of the endpoint x'_q until $\bar{c}$ has length $2^{k+1} + 1$. Notice that $\mathrm{cost}(\bar{c}) = \mathrm{cost}(c')$.

By applying the claim to $\bar{c}$, we obtain $\operatorname{cost}(c) \geq \operatorname{cost}(\bar{c}) \geq d_X(x,x') - (k+1)\delta$. Since c was arbitrary, we also have:

$$\min_{c \in C(x,x')} \operatorname{cost}(c) = u_X(x,x') \geq d_X(x,x') - (k+1)\delta.$$

Since x, x' were also arbitrary, we obtain:

$$\max_{x,x' \in X} \left(d_X(x,x') - u_X(x,x')\right) \leq (k+1)\delta \leq \log_2(2n)\operatorname{ult}(X).$$

This concludes the proof of the upper bound.

An example to show tightness. We formulate our example in terms of metric spaces. First we need to describe some constructions. Our main tool is a *metric transform*, which is a continuous, monotone increasing function $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\Psi(0) = 0$. In particular, Ψ maps metrics to metrics. For any metric space (X, d_X), we let $\Psi(X)$ denote $(X, \Psi(d_X))$. For spaces X and transforms $\Psi(X)$ such that $\operatorname{ult}(\Psi(X)) \neq 0$, we define the following quantity:

$$R(\Psi) := \frac{\|\Psi(d_X) - \Psi(u_X)\|_\infty}{\operatorname{ult}(\Psi(X))}.$$

For any $x, x' \in X$, we also define:

$$d_X^{(1)}(x,x') := \min\left\{\max\left(d_X(x,z), d_X(z,x')\right) : z \in X\right\}.$$

One can verify the following reformulation of $\operatorname{ult}(X)$:

$$\operatorname{ult}(X) = \|d_X - d_X^{(1)}\|_\infty. \qquad (2)$$

Let $0 < \varepsilon \ll 1$. Consider the *snowflake* metric transform $\Psi_\varepsilon(\alpha) = \alpha^\varepsilon$. In the limit, when $\varepsilon \to 0$, all non-zero distances would become 1. That is, $\lim_{\varepsilon \downarrow 0} \Psi_\varepsilon(X)$ would be equal to the metric space with underlying set X and the *discrete metric* (i.e. the metric attaining only the values 0 and 1). Note that the discrete metric is actually an ultrametric.

Next let X be a finite set with $n > 1$ points, E a subset of $X \times X$, such that $G = (X, E)$ becomes a connected graph with edge weights 0 (for absence of an edge) or 1 (for presence of an edge). Let (X, d_X) represent the finite metric space with n points arising from computing the graph (or path length) distance on G. Specifically,

$$d_X(x,x') := \min\{|c| : c \in C(x,x')\},$$

where $C(x,x')$ is the set of all chains connecting x and x'. In this case, u_X, the SLHC output ultrametric, will be 1 between different points. Also note that d_X takes integer values, and for any two points x, x', we have $d_X^{(1)}(x,x') = \lceil \frac{d_X(x,x')}{2} \rceil$. Such a space will be called a *graph metric space*.

Proof of tightness. For $n \geq 2$, fix $\varepsilon_n = \frac{1}{\log^2 n}$ and let $\widehat{X_n}$ denote the graph metric space on $(n+1)$ points. Note that $\operatorname{diam}(\widehat{X_n}) = n$ for each n. Next let $X_n = \Psi_{\varepsilon_n}(\widehat{X_n})$. Notice that the numerator of $R(\Psi_{\varepsilon_n})$ is now:

$$\max_{\alpha \in [0,n]} (\alpha^{\varepsilon_n} - 1^{\varepsilon_n}) = n^{\varepsilon_n} - 1.$$

By applying Equation 2, the denominator of $R(\Psi_{\varepsilon_n})$ becomes:

$$\begin{aligned}\max_{\alpha \in [0,n]} (\alpha^{\varepsilon_n} - \left\lceil \frac{\alpha}{2} \right\rceil^{\varepsilon_n}) &\approx \max_{\alpha \in [0,n]} \left(\alpha^{\varepsilon_n} - \left(\frac{\alpha}{2}\right)^{\varepsilon_n}\right) \\ &= n^{\varepsilon_n}(1 - 2^{-\varepsilon_n}) \\ &= \left(\frac{n}{2}\right)^{\varepsilon_n}(2^{\varepsilon_n} - 1).\end{aligned}$$

Notice that equality holds above for even values of n. The expression for $R(\Psi_{\varepsilon_n})$ now becomes:

$$\begin{aligned}R(\Psi_{\varepsilon_n}) = \frac{(n^{\varepsilon_n} - 1)2^{\varepsilon_n}}{n^{\varepsilon_n}(2^{\varepsilon_n} - 1)} &= \frac{n^{\varepsilon_n} - 1}{n^{\varepsilon_n}} \cdot \frac{2^{\varepsilon_n}}{2^{\varepsilon_n} - 1} \\ &= \frac{e^{\varepsilon_n \log n} - 1}{e^{\varepsilon_n \log n}} \cdot \frac{2^{\varepsilon_n}}{e^{\varepsilon_n \log 2} - 1} \\ &= \frac{e^{\frac{1}{\log n}} - 1}{e^{\frac{1}{\log n}}} \cdot \frac{2^{\varepsilon_n}}{e^{\frac{\log 2}{\log^2 n}} - 1}.\end{aligned}$$

For large n, this becomes $\approx \frac{\frac{1}{\log n}}{\frac{\log 2}{\log^2 n}} = \log_2(n) \approx \log_2(2n)$.

This proves tightness, and we conclude our proof. ☐

Additional details of the treegram construction.

From treegrams to symmetric ultranetworks. Let T_X be a treegram over X. For each $x, x' \in X$, define:

$$u_{T_X}(x,x') := \min\{t \in \mathbb{R} | \, x, x' \in X_t \text{ and } x \sim_t x'\}.$$

One can see that u_{T_X} defines a symmetric ultranetwork over X. Here symmetry follows because T_X is symmetric. We need to check the strong triangle inequality. Let $x, x', x'' \in X$. Let $t = \max(u_{T_X}(x,x''), u_{T_X}(x'',x'))$. Then $x, x', x'' \in X_t$ and $x \sim_t x'' \sim_t x'$. Thus $u_{T_X}(x,x') \leq t$.

This defines a map from treegrams to symmetric ultranetworks given by $T_X \mapsto u_{T_X}$.

Theorem 6. *Any symmetric ultranetwork has a lossless realization as a treegram, and any treegram has a lossless realization as a symmetric ultranetwork.*

More specifically, given a finite set X, let $\operatorname{Tree}(X)$ denote the set of all treegrams on X and let $\operatorname{Ultra}(X)$ denote the set of all symmetric ultranetworks on X. Next consider the maps $\Phi : \operatorname{Tree}(X) \to \operatorname{Ultra}(X)$ and $\Psi : \operatorname{Ultra}(X) \to \operatorname{Tree}(X)$ given by $T_X \mapsto u_X$ and $u_X \mapsto T_X$. Then we have $\Phi \circ \Psi = \operatorname{Id}_{\operatorname{Ultra}(X)}$ and $\Psi \circ \Phi = \operatorname{Id}_{\operatorname{Tree}(X)}$.

Proof. Let (X, ω_X) be a symmetric ultranetwork. Let $x, x' \in X$, and let $t = \omega_X(x,x')$. Then $(x,x') \in R_t$, where R_t is defined as before:

$$R_t := \{(x,x') \in X \times X : \omega_X(x,x') \leq t\}.$$

In particular, $(x,x') \notin R_s$ for $s < t$. Thus $x \sim_t x'$. So $x, x' \in X_t$ and $P_t(x) = P_t(x')$, where $(X_t, P_t) = T_X(t)$ and $T_X = \Psi(\omega_X)$. Since $(x,x') \notin R_s$ for $s < t$, it follows that $u_X(x,x') = t$, where $u_X = \Phi(T_X) = \Phi(\Psi(\omega_X))$. This holds for arbitrary $x, x' \in X$. Hence $u_X = \omega_X$, so $\Phi \circ \Psi = \operatorname{Id}_{\operatorname{Ultra}(X)}$.

Next let (X, T_X) be a treegram with subpartitions (X_t, P_t) for $t \in \mathbb{R}$, where the equivalence relation defining P_t is denoted $\sim_t$. Denote $\Phi(T_X)$ by u_X. For each $t \in \mathbb{R}$, let R_t be the relation on $X \times X$ induced by u_X. Let $X'_t = \pi_1(R_t) = \pi_2(R_t)$ and let $P'_t \in \mathrm{Part}(X'_t)$ be the partition induced by $\sim'_t$, where $x \sim'_t x'$ if and only if $(x, x') \in R_t$. Let $T'_X(t) = (X'_t, P'_t)$. Notice that $T'_X = \Psi(u_X)$.

But notice that $x \sim'_t x'$ if and only if $(x, x') \in R_t$ if and only if $x \sim_t x'$. Thus $\sim'_t$ and $\sim_t$ define the same equivalence relation for each $t \in \mathbb{R}$. It follows then that $P'_t = P_t$ for each $t \in \mathbb{R}$. We also have $X'_t = \pi_1(R_t) = X_t$. Thus $T'_X = T_X$ for all $t \in \mathbb{R}$. So $\Psi \circ \Phi = \mathrm{Id}_{\mathrm{Tree}(X)}$. This proves the result. $\square$