

## MATH 4570 – APPLIED ALGEBRAIC TOPOLOGY SYLLABUS

**Course Description.** The course will serve as an introduction to applied algebraic topology (AAT), with a view toward *persistent homology* [3] of point clouds for applications to data analysis. In order to keep the material accessible to a wide audience, an emphasis will be placed on homology of simplicial complexes over a field. We will focus on building up intuition about what homology measures through concrete examples starting from clustering methods. We will then move on to the more specialized notion of persistent homology. Real-world applications to data analysis will be provided. Theoretical ideas will be complemented with hands-on activities that will introduce students to the main software packages for implementing AAT ideas such as javaPlex [2], Ripser, [4], TDA [5], etc.

**Intended Audience and Prerequisites.** The course is designed for junior and senior undergraduate mathematics majors. The course will also be appropriate for computer science and data analytics majors with a strong math background.

The prerequisites are:

- **Discrete Math:** Example courses: Math 3345: Foundations of higher mathematics OR CSE 2321: Foundations I: Discrete Structures;
- **Linear Algebra:** MATH 2568: Linear Algebra.

In particular, no prior knowledge of topology or abstract algebra will be assumed. Students with familiarity in these subjects are welcome, as there is not a significant overlap with the standard courses.

**Grading.** Grades for the course will be determined by weekly homework assignments (60%), a midterm project (15%) and a final project (25%). Students will be given choices of topics for the midterm and final projects. These will range from guided investigations into deeper mathematics than what is covered in lecture to programming projects.

**Text.** The course will roughly follow the recent survey article [1] by Gunnar Carlsson. Background material which does not appear in the survey paper will be supplemented by additional course notes.

**Website.** <http://www.tgda.osu.edu/courses/math-4570/>

### Tentative Schedule

#### Week 1: Review of Linear Algebra I

- Course overview and motivation
- Vector spaces and subspaces over  $\mathbb{R}$  and  $F_2$
- Examples
- Basis and dimension
- Linear transformations and matrix representations
- Homework: Basic properties of vector spaces

**Week 2: Review of Linear Algebra II**

- Kernel and cokernel of a linear transformation
- Quotient vector spaces
- Inner product spaces
- Normed spaces, leading to first examples of metrics
- Homework: Calculations with linear transformations, properties of norms

**Week 3: Metric Topology I**

- Definition of a metric space
- Examples of metric spaces
- Open and closed sets
- Continuous maps between metric spaces
- Homework: Working with basic properties of metric spaces

**Week 4: Metric Topology II**

- Basic topological properties of metric spaces: connectedness, compactness
- Equivalence relations
- Homeomorphism
- How to distinguish metric spaces? Light introduction to the ideas of  $\pi_0$ ,  $\pi_1$
- Example: clustering in finite metric spaces via  $\pi_0$
- Homework: Basic topological properties

**Week 5: Homology of Simplicial Complexes I**

- Motivation: distinguishing metric spaces through linear algebra
- Return to linear algebra: free vector spaces generated by a finite set
- Homology of simplicial complexes: develop intuition by working simple examples in detail; start with calculations over  $F_2$
- Homework: Working with free vector spaces and some basic homology calculations over  $F_2$

**Week 6: Homology of Simplicial Complexes II**

- Chain complexes of vector spaces and boundary maps
- Abstract definition of homology of a chain complex of vector spaces
- Rigorous definition of homology of a simplicial complex
- Normal forms for matrix pairs as an algorithm for computing homology
- Light introduction to functoriality: inclusion maps induce maps on homology
- Homework: Calculating homology of simple examples of simplicial complexes and proving basic properties of homology

**Week 7: Homology of Simplicial Complexes III**

- Informal discussion of extending homology to general metric spaces
- Homotopy and homotopy equivalence in metric spaces
- Contractibility
- Contractible simplicial complexes have trivial homology (statement without formal proof)
- Homework: Working with homotopies

**Week 8: Point Clouds and Associated Spaces**

- Motivation: why study point clouds?
- Examples of point clouds arising from real-world data
- Point clouds as finite metric spaces
- Single-linkage clustering

- Persistent sets
- Vietoris-Rips complex
- Mid term project due

**Week 9: Persistent Homology I**

- Homology of the Vietoris-Rips complex of a point cloud
- Persistence vector spaces: definitions of persistence vector space, linear transformations, sub-persistence vector space
- Finitely-presented persistence vector spaces
- Basic properties of persistence vector spaces
- Homework: more basic properties of persistence vector spaces

**Week 10: Persistent Homology II**

- Classification theorem for finitely-presented persistence vector spaces
- Demonstration: Javaplex for topological data analysis
- Homework: filling in details of the proof of the classification theorem

**Week 11: Persistence Diagrams**

- Barcodes and persistence diagrams
- Persistent homology algorithm
- Computational examples
- Examples and applications of barcodes in the literature

**Week 12: Structures on the Space of Barcodes I**

- Define bottleneck distance on Barcode space
- Define Gromov-Hausdorff distance on the space of finite metric spaces
- Discuss the stability theorem relating the two distances (without proof)
- Homework: fill in details showing the bottleneck distance and Gromov-Hausdorff distance are metrics

**Week 13: Structures on the Space of Barcodes II**

- Define interleaving distance
- Work with a variety of simple examples to develop intuition about interleaving distance
- Sketch the proof of the isometry theorem relating interleaving distance to bottleneck distance
- Homework: fill in some details of the proof of the isometry theorem

**Week 14: Applications**

- The last week will be spent studying specific applications to real-world data. This can be catered to interests of the students.
- Final project

**REFERENCES**

- [1] Carlsson, G., 2014. Topological pattern recognition for point cloud data. Acta Numerica, 23, p.289.
- [2] <http://appliedtopology.github.io/javaplex/>
- [3] [https://en.wikipedia.org/wiki/Persistent\\_homology](https://en.wikipedia.org/wiki/Persistent_homology)
- [4] <https://github.com/Ripser/ripser>
- [5] <https://cran.r-project.org/web/packages/TDA/index.html>